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COMMENT

A comment on the solutions of the equation $\nabla \times \boldsymbol{a} = k\boldsymbol{a}$

V K Varadan, A Lakhtakia and V V Varadan

The Center for the Engineering of Electronic and Acoustic Materials and Department of Engineering Science and Mechanics, The Pennsylvania State University, University Park, PA 16802, USA

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Abstract. It is pointed out here that solenoidal solutions of the equation $\nabla \times \boldsymbol{a} = k\boldsymbol{a}$ exist in magnetoelectric media, in which case $\boldsymbol{a}$ must be identified as a linear combination of the electric and the magnetic fields.

In two recent communications, Salingaros (1986a, b) has claimed that the solutions of the equation

$$\nabla \times \boldsymbol{a} = k\boldsymbol{a} \tag{1}$$

do not correspond to physical magnetic fields, despite their widespread use in both plasma physics and astrophysics as force-free magnetic fields. This conclusion is based upon the transformation properties of this equation under (i) gauge transformations, (ii) parity and (iii) change of basis. Issue, however, with this conclusion has been taken by Maheswaran (1986) who has shown that a non-vanishing magnetic field which must satisfy Maxwell's equations everywhere can satisfy the force-free condition (1) in some restricted domain.

Without commenting on the merits of the two sides of this controversy regarding force-free magnetic fields, we wish to point out here that solenoidal solutions of equation (1) exist and are very physical when dealing with magnetoelectric media. However, in that case, $\boldsymbol{a}$ must be identified as a linear combination of the electric and the magnetic fields.

Consider a region V occupied by an isotropic magnetoelectric medium in which the usual constitutive relations $\boldsymbol{D} = \varepsilon\boldsymbol{E}$ and $\boldsymbol{B} = \mu\boldsymbol{H}$ do not hold, due to their incompatibilty with the handedness of the medium. Instead, the relations

$$\boldsymbol{D} = \varepsilon\boldsymbol{E} + \alpha\varepsilon\nabla \times \boldsymbol{E} \qquad \boldsymbol{B} = \mu\boldsymbol{H} + \beta\mu\nabla \times \boldsymbol{H} \tag{2}$$

hold (e.g., Bohren and Huffman 1983). Use is now made of the usual Maxwell's equations, along with equation (2), to obtain the differential relations

$$\nabla^2\begin{bmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{bmatrix} = -[K]^2\begin{bmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{bmatrix} \qquad \nabla \times \begin{bmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{bmatrix} = [K]\begin{bmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{bmatrix} \qquad \nabla \cdot \begin{bmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} \tag{3}$$

where the matrix

$$[K] = (1 - k^2\alpha\beta)^{-1}\begin{bmatrix}k^2\beta & \mathrm{j}\omega\mu\\ -\mathrm{j}\omega\varepsilon & k^2\alpha\end{bmatrix} \qquad k = \omega(\mu\varepsilon)^{1/2} \tag{4a, b}$$

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and an $\exp[-j\omega t]$ time dependence has been assumed. The electromagnetic field is now transformed to

$$\begin{bmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{bmatrix} = [A] \begin{bmatrix} \boldsymbol{a}_L \\ \boldsymbol{a}_R \end{bmatrix} \tag{5}$$

where the left- and right-circularly polarised fields, $\boldsymbol{a}_L$ and $\boldsymbol{a}_R$, respectively, must satisfy the vector Helmholtz equation:

$$\{\nabla^2 + k_L^2\}\boldsymbol{a}_L = 0 \qquad \{\nabla^2 + k_R^2\}\boldsymbol{a}_R = 0 \tag{6a, b}$$

along with the auxiliary conditions

$$\nabla \times \boldsymbol{a}_L = k_L \boldsymbol{a}_L \qquad \nabla \cdot \boldsymbol{a}_L = 0 \tag{7a, b}$$

$$\nabla \times \boldsymbol{a}_R = -k_R \boldsymbol{a}_R \qquad \nabla \cdot \boldsymbol{a}_R = 0. \tag{8a, b}$$

In these equations, the matrix

$$[A] = \begin{bmatrix} 1 & A_R \\ A_L & 1 \end{bmatrix} \tag{9}$$

while

$$k_L = k(1 - k^2\alpha\beta)^{-1}\{[1 + (\alpha - \beta)^2 k^2/4]^{1/2} + (\alpha + \beta)k/2\} \tag{10a}$$

$$k_R = k(1 - k^2\alpha\beta)^{-1}\{[1 + (\alpha - \beta)^2 k^2/4]^{1/2} - (\alpha + \beta)k/2\} \tag{10b}$$

$$j\omega\varepsilon A_R = k_R(1 - k^2\alpha\beta) + k^2\alpha \tag{10c}$$

$$j\omega\mu A_L = k_L(1 - k^2\alpha\beta) - k^2\beta \tag{10d}$$

the real parts of both k_L and k_R being positive.

Thus it has been shown that solenoidal solutions $\boldsymbol{a}_L$ and $\boldsymbol{a}_R$ of equation (1) exist which satisfy Maxwell's equations in magnetoelectric media. Physically, these solutions represent circularly polarised fields of different helicities and are specific linear combinations of the electric and the magnetic fields (see equation (5)).

References

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